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CONCERNING SOME SPACECRAFT CONVERGENCE CONTROL LAWS

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The problem of motion of an interceptor spacecraft along a three-dimensional trajectory in a central gravitational field is considered; this trajectory is the mapping in the involute plane of the shape, dimensions, and orientation of the Keplerian orbit of the target spacecraft. Control laws which yield analytical solutions of the encounter problem are chosen. The active spacecraft is referred to as the "interceptor", the passive spacecraft as the "target".

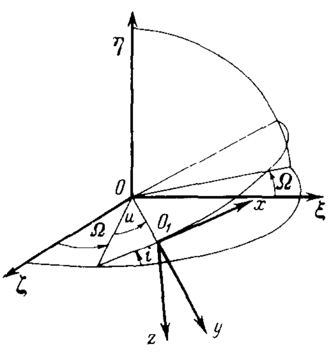

Fig. 1

1. The motion of the interceptor under the controlling acceleration W applied to its center of mass O_1 is described by equations in the rotating right-hand orthogonal coordinate system $Oxyz$ whose y-axis coincides with the radius vector constructed from the attracting center O to the point O_1, and whose x-axis coincides with the direction of motion in such a way that the vector of the absolute velocity of the interceptor's center of mass lies in the xy-plane. The orientation of the axes xyz relative to the inertial coordinates is defined (see Fig. 1) by the longitude Ω of the ascending node, the inclination i of the instantaneous orbital plane to the equator, and the range angle u. The equations of motion of the center of mass of the interceptor are

$$\dot{V}_x = W_x + \omega_z V_y, \qquad \dot{V}_y = W_y - \omega_z V_x - g \tag{1.1}$$
$$0 = W_z + \omega_y V_x, \quad \omega_z = -V_x / r, \; g = g_0 (R_0 / r)^2$$

The rates of change of the angles defining the orientation of the rotating axes relative to the inertial axes are given by the differential equations

$$\frac{d\Omega}{dt} = \omega_y \frac{\sin u}{\sin i}, \quad \frac{di}{dt} = \omega_y \cos u, \quad \frac{du}{dt} = -\omega_z - \omega_y \sin u \operatorname{ctg} i \tag{1.2}$$

We shall make our choice of the control law for the motion of the center of mass of the interceptor subject to the conditions of integrability of equations of motion (1.1), (1.2); moreover, we shall restrict its choice to the class of functions in which the control constants ensuring convergence of the spacecraft can be determined with sufficient ease.

We showed in [1] that kinematic equations (1.2) are integrable regardless of the control law for the motion of the center of mass of the interceptor in the involute plane if the projection of the controlling acceleration on the direction perpendicular to the instantaneous orbital plane varies according to the law

$$W_z = KV_x^2 / r \qquad (K = \text{const}) \tag{1.3}$$

The presence of a controlling acceleration W_z implies motion of the orbital plane of the interceptor. By suitable choice of the quantity K in control law (1.3) and of the instant of its actuation we can ensure coincidence of the orbital planes of the interceptor and target. This is the control law for the motion of the orbital plane which we adopt in the present paper.

The motion of the interceptor in the involute plane along a trajectory having the shape and dimensions of the target's orbit is subject to the constraints

$$\operatorname{tg} \theta = \pm \sqrt{ar^2 + br - 1}, \quad a = \frac{2h}{C^2}, \quad b = \frac{2g_0R_0^2}{C^2} \tag{1.4}$$

$$r_{pk} = r_{qk} \qquad \text{for } t = t_k$$

Here h is the constant of the energy integral, C is the constant of the interceptor-target area integral, the subscripts p and q denote the interceptor and target, respectively, and θ is the angle of inclination of the absolute-velocity vector V to the local horizon. This constraint was obtained by transforming the energy and interceptor-target area integrals. The first two equations of (1.1) describe the motion of the interceptor in the involute plane. The polar angle between the initial and final positions of the radius vector of the interceptor's center of mass in this plane is given by

$$J = -\int_{t_0}^{t_k} \omega_z dt \tag{1.5}$$

Knowing this angle, we can determine the required value of r_{p0} at the start of the controlled motion which ensures fulfilment of the second condition of (1.4), and also the angle θ_{p0}.

In fact,

$$V_y = V_x \operatorname{tg}\theta \qquad (V_y = r^\cdot) \tag{1.6}$$

and, recalling (1.4) and the fourth equation of (1.1), we obtain

$$J = \int_{r_{l0}}^{r_{pk}} \frac{dr}{r\sqrt{ar^2 + br - 1}} \qquad \text{for } \Delta = -4a - b^2 \neq 0 \tag{1.7}$$

The angle J in the segment of the interceptor's trajectory where $W_z = 0$ must be determined from the initial conditions. Over the segment where $W_z \neq 0$ the angle J is given by the formulas [1]

$$J = \int_{x_0}^{x_k} \frac{dx \operatorname{sign}(K \cos u)}{\sqrt{-(1+K^2)x^2 + 2kx + K^2 - k^2}} \qquad (x = \cos i)$$

$$\cos i - K \sin u \,\sin i = k, \qquad k = \cos i_0 - K \sin u_0 \sin i_0 \tag{1.8}$$

$$\Omega_k - \Omega_0 = \int_{x_0}^{x_k} \frac{(x-k)\operatorname{sign}(K\cos u)\ dx}{(x^2-1)\sqrt{-(1+K^2)x^2 + 2kx + K^2 - k^2}}$$

The quantities K and k can be determined by substituting the given values of Ω_0, i_0,

Ω_k, i_k, u_k into Eqs. (1.8), where the subscripts 0 and k denote the start and termination of the maneuver of rotating the orbital plane until it coincides with the orbital plane of the target.

The resulting value of the integration constant k can be used to find the range angle u_0 whose attainment coincides with the start of controlled motion of the interceptor's plane.

Thus, if the time interval between the beginning and end of controlled motion in the involute plane is larger than the time interval between the beginning and end of the orbit rotation maneuver, i.e. if the former interval includes the latter, then the angle J breaks down into three parts on a unit sphere; the motion over the first and last parts is along great-circle arcs; the motion over the middle part of the angle proceeds along the curve defined by Eqs. (1.8). The curve is a minor-circle arc.

2. The first equation of (1.1) is integrable if

$$W_x = f(t) / r \tag{2.1}$$

The structure of $f(t)$ will be determined when we synthesize the control laws (see below). Integrating this equation, we obtain

$$V_x r = V_{x0} r_0 + \int_{t_0}^{t} f(\tau)\, d\tau = \varphi(t) \tag{2.2}$$

Integrating Eq. (1.6) with allowance for (2.2) and the first equation of (1.4), we obtain

$$\int_{r_{p0}}^{r_p} \frac{r dr}{\pm \sqrt{ar^2 + br - 1}} = \int_{t_0}^{t} \varphi(\tau)\, d\tau = \varphi_1(t) \tag{2.3}$$

The set of equations (1.4)–(1.8), (2.2), (2.3) completely defines the motion of the interceptor in time. The next step is to find the law for the controlling acceleration W_y which ensures fulfilment of constraint (1.4). Transforming (1.6) with allowance for (1.4) and (2.2), we obtain

$$V_y = \pm \frac{\varphi(t)}{r} \sqrt{ar^2 + br - 1} \tag{2.4}$$

Differentiating this equation and combining the result with the second equation, we obtain the general form of the required control law,

$$W_y = \frac{f(t)}{\varphi(t)} r^{\cdot} + g\left(1 - \frac{\varphi^2(t)}{C^2}\right) \tag{2.5}$$

3. Synthesis of the structure of control laws (2.1) and (2.5) begins with determination of the conditions ensuring contact of the interceptor and target with a prescribed ratio of their velocities V_{pk}/V_{qk}. Transforming Eqs. (2.2) and (2.4), we obtain

$$V_{pk} / V_{qk} = \varphi(t_k) / C, \quad V_{qk} = C \sqrt{ar_k^2 + br_k} / r_k \tag{3.1}$$

Here V_{qk} is the velocity of the center of mass of the target at the point of contact. Since $\theta_q = \theta_p$ at this point, it follows that

$$V_{xp} / V_{xq} = V_{yp} / V_{yq} = \varphi(t_k) / C$$

Equation (3.1) yields the first condition

$$\varphi(t_k) = C_1 \tag{3.2}$$

which the required control laws must satisfy. According to (2.3) the second condition is

$$\varphi_1(t_k) = C_2 \tag{3.3}$$

Relations (3.2), (3.3) imply that the function $f(t)$ must depend on at least two parameters, i. e. $f(t) = f(\alpha_0, \alpha_1, t)$.

In the simplest case where the target is a satellite moving in a circular orbit and control of the motion of the interceptor's orbital plane coincides in time with control in the involute plane, the function $f(t)$ can be found from the minimum condition for the functional

$$J = \int_{t_0}^{t_k} W^2 dt, \quad W^2 = W_x^2 + W_y^2 + W_z^2 \qquad \text{for } G = \int_{t_0}^{t_k} \varphi(t)\, dt = C_2 \tag{3.4}$$

Differentiation of (2. 2) yields the relation

$$\varphi^{\cdot}(t) = f(t) \tag{3.5}$$

For $r = \text{const}$ we have $C^2 = gr^3$, and the second equation of (3. 4) with allowance for (2. 1), (2. 5), (1. 3) and (3. 5) becomes

$$W^2 = (\varphi^{\cdot 2} + a_1\varphi^4 + a_2\varphi^2 + a_3) / r^{\cdot 2}, \quad a_1 = (K^2 + 1)/r^4, \quad a_2 = -2g/r, \quad a_3 = g^2 r^2 \tag{3.6}$$

Thus, Eqs. (3. 4) with allowance for (3. 6) reduce to the isoperimetric problem of variational calculus in which the required control is an extremum of the integral

$$J_1 = \int_{t_0}^{t_k} (W^2 + \lambda\varphi)\, dt = \min \qquad (\lambda = \text{const}) \tag{3.7}$$

Integrating the Euler equation for functional (3. 7) with allowance for (2. 1) and (2. 2), we obtain

$$\varphi^{\cdot 2} = a_1\varphi^4 + a_2\varphi^2 + \lambda r^2\varphi + W_{x0}^2 r^2 - (K^2 + 1)V_0^4 + 2grV_0^2 - \lambda V_0 r^3$$

This equation reduces to the quadrature

$$\int_{\varphi_0}^{\varphi} \frac{dy}{\pm\sqrt{y^4 + b_1 y^2 + b_2 y + b_3}} = \frac{(t - t_0)\sqrt{K^2 + 1}}{r^2}, \qquad b_1 = -\frac{2gr^3}{K^2 + 1}$$
$$b_2 = \frac{\lambda r^6}{K^2 + 1}, \qquad b_3 = \frac{r^4}{K^2 + 1}\left[W_{x0}^2 r^2 - (K^2 + 1)V_0^4 + 2grV_0^2 - \lambda V_0 r^3\right] \tag{3.8}$$

The form of the solution of Eq. (3. 8) depends on the roots of the equation

$$y^4 + b_1 y^2 + b_2 y + b_3 = 0$$

However, these equations can be expressed in terms of elliptical integrals of the first kind in every case. Inversion of the elliptic integral yields the function $\varphi(t)$, which is a third-degree polynomial when two terms are retained in the expansion.

The parameters λ and W_{x0} can be determined from boundary conditions (3. 2), (3. 3). If it is necessary to find the control of simplest mathematical structure, them we can take

$$f(t) = \alpha_0 + \alpha_1 t \tag{3.9}$$

Then, recalling (2. 2) and (2. 3), we obtain

$$\varphi(t) = \alpha_0 t + {}^1\!/_2\alpha_1 t^2 + \alpha_2 \quad (\alpha_2 = V_{x0} r_0)$$
$$\varphi_1(t) = {}^1\!/_2\alpha_0 t^2 + {}^1\!/_6\alpha_1 t^3 + \alpha_2 t \tag{3.10}$$

for $t_0 = 0$ The control parameters α_0 and α_1 can be found from Eqs. (3. 10) for $t = t_k$ with allowance for boundary conditions (3. 2) and (3. 3). Having determined the coefficients α_0 and α_1, we assume that we have determined control laws (1. 3), (2. 1) and (2. 5), provided the absolute value of the controlling acceleration W satisfies the inequality

$$W_{\min}(t) \leqslant W(t) \leqslant W_{\max}(t)$$

If this inequality is not fulfilled, then the final conditions (i. e. the neighborhood and time of interception) for given initial conditions are altered; conversely, the initial conditions are altered for given final conditions.

Control law (3.9) is valid for interception of a target in any orbit, be it circular, elliptical, parabolic, or hyperbolic.

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CRITERION OF EXISTENCE OF AN OPTIMAL CONTROL FOR A CLASS OF LINEAR STOCHASTIC SYSTEMS

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We consider a control system described by an nth order differential equation with random coefficients. Necessary and sufficient conditions of existence of a linear control stabilizing such a system in the mean square and conveying a minimum to the quadratic quality criterion are obtained. The problem of stabilization of a stochastic system in which the noise depends on the magnitude of the controlling force was also studied in [1].

1. Let a linear stochastic system be given, defined by the following nth order differential equation:

$$y^{(n)} + [a_1 + \xi_1^{\cdot}(t)]y^{(n-1)} + \ldots + [a_n + \xi_n^{\cdot}(t)]y = [b + \sigma\eta^{\cdot}(t)]u \quad (1.1)$$

where

$$a_i = \text{const}, \qquad b_i = \text{const} \neq 0 \qquad (i = 1,2,\ldots,n)$$

u is a scalar control, $\xi_i^{\cdot}(t)$ are the Gaussian white noises with zero mathematical expectation which are, in general, interrelated in such a way that

$$M\xi_i^{\cdot}(t)\,\xi_j^{\cdot}(s) = 2a_{ij}\delta(t-s)$$

and $\eta^{\cdot}(t)$ is a white noise process independent of the set $\xi_1^{\cdot}(t),\ldots,\xi_n^{\cdot}(t)$. In addition

$$M\eta^{\cdot}(t) = 0, \qquad M\eta^{\cdot}(t)\eta^{\cdot}(s) = 2\delta(t-s)$$

Let us set

$$y = X_1, \quad y' = X_2,\ldots,y^{(n-1)} = X_n$$

Then (1.1) can be assumed to represent a system of stochastic differential Ito equations (see e.g. [2])

$$dX_1 = X_2dt, \quad dX_2 = X_3dt,\ldots,dX_{n-1} = X_ndt \qquad (1.2)$$

$$dX_n = \Big(-\sum_{i=1}^{n} a_iX_{n-i+1} + bu\Big)dt - \sum_{i,\,j=1}^{n} \alpha_{ij}X_{n-i+1}\,d\eta_j(t) + \sigma u\,d\eta(t)$$

where $\eta_1, \eta_2,\ldots,\eta_n$ and η denote mutually independent Gaussian Markov processes for which

$$M\eta^{\cdot}_i(t) = 0, \qquad M\eta_i^2(t) = 2t$$

and the matrix $\|\alpha_{ij}\|$ is obtained from the condition

$$\|\alpha_{ij}\|\,\|\alpha_{ji}\| = \|a_{ij}\|$$